F2 June 2018 (MA)

Q1)

$$\frac{1}{x-2} > \frac{2}{x}$$

$\times (x-2)^2$

$$\frac{(x-2)^2}{(x-2)} > \frac{2(x-2)^2}{x}$$

$\times x^2$

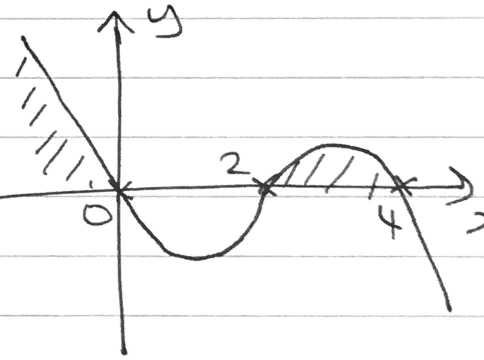
$$x^2(x-2) > 2x(x-2)^2$$

$$x(x-2)\left[x - 2(x-2)\right] > 0$$

$$x(x-2)(4-x) > 0$$

critical values: $x=0$, $x=2$, $x=4$

Negative x^3 shape ...



we want where $y>0$

so ...

$$\boxed{\begin{array}{c} x < 0 \\ 2 < x < 4 \end{array}}$$

Q2a) $(x^2+1)\frac{dy}{dx} + xy = x$

$\div (x^2+1)$: $\frac{dy}{dx} + \frac{xy}{x^2+1} = \frac{x}{x^2+1}$

$I = e^{\int \frac{x}{x^2+1}dx} = e^{\frac{1}{2}\ln(x^2+1)} = (x^2+1)^{1/2}$

By Pattern

$\div (x^2+1)^{\frac{1}{2}}$: $(x^2+1)^{\frac{1}{2}}\frac{dy}{dx} + \frac{xy}{(x^2+1)^{1/2}} = \frac{x}{(x^2+1)^{1/2}}$

$\therefore \frac{d}{dx}\left(y(x^2+1)^{1/2}\right) = \frac{x}{(x^2+1)^{1/2}}$

$y(x^2+1)^{1/2} = \int \frac{x}{(x^2+1)^{1/2}}dx$ By pattern

$y(x^2+1)^{1/2} = \frac{1}{2}\cdot\frac{(x^2+1)^{1/2}}{1/2} + c$

$\therefore y(x^2+1)^{1/2} = (x^2+1)^{1/2} + c$

$\div (x^2+1)^{1/2}$: $\boxed{y = 1 + \frac{c}{\sqrt{x^2+1}}}$

b) $\underline{y=2, x=3}$: $2 = 1 + \frac{c}{\sqrt{10}}$

$1 = \frac{c}{\sqrt{10}}$ $\therefore$ $c = \sqrt{10}$ //

$\therefore$ $\boxed{y = 1 + \frac{\sqrt{10}}{\sqrt{x^2+1}}}$

Q3a) $\frac{d}{dx}[2y'' + y' - xy = 1]$

$\Rightarrow 2y''' + y'' - y - xy' = 0$

$\frac{d}{dx}[2y''' + y'' - y - xy'] = 0$

$\Rightarrow 2y'''' + y''' - y' - y' - xy'' = 0$

$\Rightarrow 2y'''' = 2y' + xy'' - y'''$

$\Rightarrow y'''' = \frac{d^4y}{dx^4} = \frac{1}{2}(2y' + xy'' - y''') = \frac{1}{2}\left(2\frac{dy}{dx} + x\frac{d^2y}{dx^2} - \frac{d^3y}{dx^3}\right)$

$a = 2$
$b = 1$
$c = -1$

b) $y=1, y'=1, x=2$

using given eqn : $2y'' + 1 - 2 = 1$

$\therefore y'' = 1$

from a : $2y''' + y'' - y - xy' = 0$

$y''' = \dfrac{xy' + y' - y''}{2} = \dfrac{2 + 1 - 1}{2} = 1$

from a : $y''' = \frac{1}{2}\left(2y' + xy'' - y'''\right) = \frac{1}{2}\left(2 + 2 - 1\right) = \frac{3}{2}$

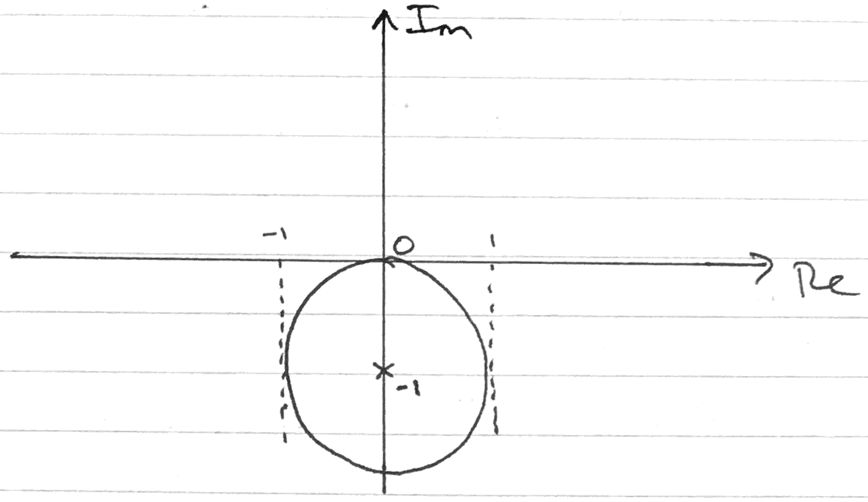
so... $y \simeq 1 + (x-2) + \dfrac{(x-2)^2}{2!} + \dfrac{(x-3)^3}{3!} + \dfrac{\frac{3}{2}(x-4)^4}{4!}$

$\Rightarrow y \simeq 1 + (x-2) + \dfrac{(x-2)^2}{2} + \dfrac{(x-2)^3}{6} + \dfrac{(x-2)^4}{16}$

c) $x = 2.1$: $y \simeq 1 + 0.1 + \dfrac{0.1^2}{2} + \dfrac{0.1^3}{6} + \dfrac{0.1^4}{16}$

$\simeq \boxed{1.105}$

Q4a) $|z+i| = 1$ $\rightsquigarrow$ circle centre $(0,-1)$ with radius 1.



b) We are given information about the z-plane ie ($|z+i|=1$) so make z the subject...!

$$w = \frac{3iz - 2}{z+i}$$

$$zw + iw = 3iz - 2$$

$$z(w-3i) = -2 - iw$$

$$z = \frac{-2-iw}{w-3i} = \frac{2+iw}{3i-w} \quad //$$

+i to each side : $z + i = \dfrac{2+iw}{3i-w} + i$

$$z + i = \frac{2+iw}{3i-w} + \frac{i(3i-w)}{3i-w}$$

$$z+i = \frac{2+iw-3-wi}{3i-w} = \frac{-1}{3i-w}$$

$$|z+i| = \frac{|-1|}{|3i-w|}$$

$$|z+i| = 1 = \frac{1}{|w-3i|}$$

$$\therefore \quad 1 = \frac{1}{|w-3i|}$$

so $|w-3i| = 1$

$$|u+iv-3i| = 1$$

$$|u+i(v-3)| = 1$$

$$u^2+(v-3)^2 = 1^2$$

$$u^2+(V-3)^2 = 1$$

Q5a) $\dfrac{4r+2}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

$4r + 2 = A(r+1)(r+2) + B(r)(r+2) + C(r)(r+1)$

$\underline{r=0}$: $2 = 2A$ $\therefore A = 1$ //

$\underline{r=-1}$: $-2 = -B$ $\therefore B = 2$ //

$\underline{r=1}$: $\cancel{6} = 6\cancel{(1)} + 2(3) + C(2)$

$\therefore C = -3$ //

so $\dfrac{4r+2}{r(r+1)(r+2)} = \dfrac{1}{r} + \dfrac{2}{r+1} - \dfrac{3}{r+2}$ //

b) $\sum_{r=1}^{n} \frac{1}{r} + \frac{2}{r+1} - \frac{3}{r+2}$ . . . -

$n=1$: $\frac{1}{1} + \frac{2}{2} - \frac{3}{3}$

Blue indicates terms that are cancelling out.

$n=2$: $\frac{1}{2} + \frac{2}{3} - \frac{3}{4}$

$n=3$: $\frac{1}{3} + \frac{2}{4} - \frac{3}{5}$

. . . .

$n=n-1$: $\frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+1}$

$n=n$: $\frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2}$

$$\Rightarrow \frac{1}{1} + \frac{2}{2} + \frac{1}{2} - \frac{3}{n+1} + \frac{2}{n+1} - \frac{3}{n+2}$$

$$\Rightarrow \frac{5}{2} + \frac{-1}{n+1} - \frac{3}{n+2}$$

$$\Rightarrow \frac{5(n+1)(n+2) - 2(n+2) - 3(2)(n+1)}{2(n+1)(n+2)}$$

$$= \frac{5(n^2+3n+2)-2n-4-6n-6}{2(n+1)(n+2)}$$

$$= \frac{5n^2+15n+10-8n-10}{2(n+1)(n+2)}$$

$$= \frac{5n^2+7n}{2(n+1)(n+2)} = \boxed{\frac{n(5n+7)}{2(n+1)(n+2)}}$$

Q6a)

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = x^2$$

$$x = e^t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt}$$

$$\therefore \frac{dy}{dx} = e^{-t}\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right) \times \frac{dt}{dx}$$

$$\left(\frac{dt}{dx} = e^{-t}\right)$$

So $\frac{d^2y}{dx^2} = -e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2}$

<u>subbing $\frac{dy}{dx} \Big/ \frac{d^2y}{dx^2}$:</u>

$$x^2\left(-e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2}\right) - 3x\left(e^{-t}\frac{dy}{dt}\right) + 3y = x^2$$

$x = e^t \ldots$

$$\Rightarrow e^{2t}\left(-e^{-2t}\frac{dy}{dt} + e^{-2t}\frac{d^2y}{dt^2}\right) - 3e^t\left(e^{-t}\frac{dy}{dt}\right) + 3y = e^{2t}$$

$$\Rightarrow -\frac{dy}{dt} + \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 3y = e^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{2t}$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = e^{2t}$$

b) AUX : $\lambda^2 - 4\lambda + 3 = 0$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3, \quad \lambda = 1 \;//$$

C.F : $y = Ae^{3t} + Be^{t}$

P.I : let $y = \lambda e^{2t}$

then $\dot{y} = 2\lambda e^{2t}$

and $\ddot{y} = 4\lambda e^{2t}$

Subbing back :

$$4\lambda e^{2t} - 8\lambda e^{2t} + 3\lambda e^{2t} = e^{2t}$$

comparing coefficients to find λ :

$$4\lambda - 8\lambda + 3\lambda = 1$$

$$\therefore -\lambda = 1 \quad \text{so} \quad \lambda = -1 \;//$$

General solution : $\boxed{y = Ae^{3t} + Be^{t} - e^{2t}}$

$x = e^t \longrightarrow \ln x = t$

$\therefore\ y = Ae^{3\ln x} + Be^{\ln x} - e^{2\ln x}$

$$\boxed{y = Ax^3 + Bx - x^2}$$

Q7a) $(\cos\theta + i\sin\theta)^7 = c^7 + 21c^5(is)^2 + 35c^3(is)^4 + 7(is)^6(c)$

$c = \cos\theta$

$s = \sin\theta$

(considering only real terms)

$\therefore\ \cos 7\theta = c^7 - 21c^5s^2 + 35c^3s^4 - 7s^6c$

$= c^7 - 21c^5(1-c^2) + 35c^3(1-c^2)^2 - 7c(1-c^2)^3$

$= c^7 - 21c^5 + 21c^7 + 35c^3 - 35(2c^5) + 35c^7 - 7c(1-2c^2+c^4)(1-c^2)$

$= c^7(22) + 35c^7 - 21c^5 + 35c^3 - 70c^5 - 7c(1 - c^2 - 2c^2 + 2c^4 + c^4 - c^6)$

$= 57c^7 - 21c^5 + 35c^3 - 70c^5 - 7c + 21c^3 - 21c^5 + 7c^7$

$= 64c^7 - 21c^5 - 21c^5 + 56c^3 - 70c^5 - 7c$

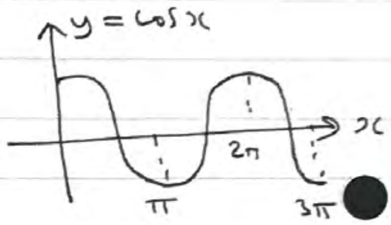
$= 64c^7 - 112c^5 + 56c^3 - 7c$

b) let $\cos\theta = x$,

then $\cos 7\theta = 64x^7 - 112x^5 + 56x^3 - 7x$

if $64x^7 - 112x^5 + 56x^3 - 7x + 1 = 0$

then $\cos 7\theta + 1 = 0$



$\cos 7\theta = -1$

$7\theta = \cos^{-1}(-1) = \pi$ //

$7\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi, \ldots$

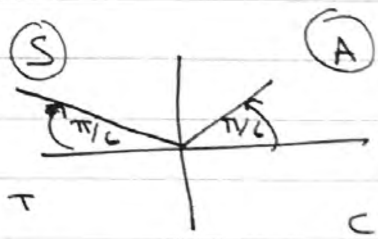
$\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \ldots$

so $x = \cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}, \cos\pi, \cos\frac{9\pi}{7} \ldots$

$\boxed{x = 0.901, 0.223, -0.623, -1}$

repeated solution.

(Q8a) $2\sin\theta = 1.5 - \sin\theta$



$3\sin\theta = 1.5$

$\sin\theta = \frac{1}{2} \longrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ //

$r = 1.5 - \frac{1}{2} = 1$ (for P and Q).

so $\boxed{P\left(1, \frac{\pi}{6}\right)}$ and $\boxed{Q\left(1, \frac{5\pi}{6}\right)}$

b) $R = 2 \times \frac{1}{2} \int_0^{\pi/6} [2\sin\theta]^2 \, d\theta + \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [1.5 - \sin\theta]^2 \, d\theta$

$R = \int_0^{\pi/6} [4\sin^2\theta] \, d\theta + \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[\frac{9}{4} - 3\sin\theta + \sin^2\theta\right] d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$=$ $\therefore \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$R = 4 \int_0^{\pi/6} \left[\frac{1}{2} - \frac{1}{2}\cos 2\theta\right] d\theta + \frac{1}{2} \int_{\pi/6}^{5\pi/6} \left[\frac{9}{4} + \frac{1}{2} - 3\sin\theta - \frac{1}{2}\cos 2\theta\right] d\theta$

$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} + \frac{1}{2}\left[\frac{11\theta}{4} + 3\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$

$= 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] + \frac{1}{2}\left[\frac{55\pi}{24} - \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{8}\right]$

$- \frac{1}{2}\left[\frac{11\pi}{24} + \frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right]$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} - \frac{11\sqrt{3}}{16} + \frac{55\pi}{48} - \frac{11\pi}{48} - \frac{3\sqrt{3}}{4}$$

$$+ \frac{\sqrt{3}}{16}$$

$$= \boxed{\frac{5\pi}{4} - \frac{15\sqrt{3}}{8}}$$